Discrete Analogue of the Verhulst Equation and Attractors. Methodological Aspects of Teaching

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Abstract. Verhulst equation in differential and discrete form is very important in different fields of material science, sociology and economics. Methodological aspects of teaching, when both models are presented for solving advanced tasks, are considered. Solutions presented in graphical and numerical forms are discussed in the framework of attractors.

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Introduction

Logistics is the art of computing. In the middle of the XIX century, Belgian mathematician Pierre François Verhulst studied the population growth. He established that initial stage of growth is approximately exponential; then, as saturation begins, the growth slows, and at the maturity, growth stops. In 1838 Verhulst introduced the logistic equation with a maximum value for the population (partial logistic growth model) [1]. A typical application of the logistic equation is a *common model of population growth*.

Applications of logistics function are useful in many fields, including material science (chemistry, geoscience), bioscience (biomathematics, artificial neural networks, ecology), sociology (political science, mathematical psychology, demography), economics (spreading of innovations, finance), linguistics (machine learning).

Presented topic is included in the master study course of *Applied mathematical methods* in the study programme *Information systems* (ISMA, Riga, Latvia).

This work is devoted to estimation of the methodological aspects of teaching, when differential and discrete models for solving the advanced tasks are presented. Moreover, the profit of attractors is discussed in student-friendly manner.

1. Literature review

Traditional estimation of dynamical systems is described by Robinson [2]. Several useful mathematical methods based on iterations allow estimating the one-dimensional dynamics and describing the chaos as a determined system. Also, Strogatz [3] represents an overview of mentioned systems for practical needs in natural sciences. Pearl [4] analyses the cause and effect relations which are fundamentally deterministic. He pointed out that cause and effect analysis must be estimated using probability factor.

Generally, conception of chaos was presented and analysed by Holmgren [5], Kinnunen [6], Alligood [7]. Peitgen [8] describes several types of attractors. Muray [9] analyses the biological oscillators. Kapica et al. [10] represent the complicated structures with bifurcational behaviour.

2. Logistic growth model

The main idea was formulated by Verhulst [1]: the rate of reproduction is proportional to both the existing population and the amount of available resources (all else being equal).

Let x=$x(t)$ represents the population size at time t, when k is the maximum possible population size (the capacity of the environment), x∈[0;k]. Two initial assumptions for deriving the equation are presented below.

1. The rate of reproduction of the population is proportional to its current value x.
2. The rate of reproduction of the population is proportional to the amount of available resources which, in turn, is proportional to the value $(k{-}x)$:

$$k - x = k\left(1 - \frac{x}{k}\right). \tag{1}$$

Note that fight for resources limits the growth of the population.

The rate of reproduction is the derivative of x with respect to t. The equation can be represented in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{k}\right), \tag{2}$$

where parameter r>0 represents the coefficient of proportionality characterizing the rate of population growth. We will consider different values of r. The initial population size is given by the initial condition: $x(0)$=x_0.

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Solution of equation. Note that the functions x=0 and x=k for all t are solutions of this equation, so we solve the equation for x∈(0;k). Making the substitutions

$$y=\frac{x}{k},\quad x=ky,\quad \frac{\mathrm{d}x}{\mathrm{d}t}=k\frac{\mathrm{d}y}{\mathrm{d}t}, \tag{3}$$

where

$$y=y(t),\quad y(0)=y_0,\quad y_0=\frac{x_0}{k}, \tag{4}$$

we get the following equations for y∈(0;1):

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ry(1-y) \tag{5}$$

or in differential form:

$$dy=ry(1-y)dt. \tag{6}$$

Eq.(6) represents the equation with separable variables. After separation

$$\frac{\mathrm{d}y}{y(1-y)}=r\mathrm{d}t,\qquad (y\neq 0,\quad y\neq 1), \tag{7}$$

we integrate it in timescale from 0 to t:

$$\int\limits_{y_0}^{y}\frac{\mathrm{d}y}{y(1-y)}=\int\limits_{0}^{t}r\mathrm{d}t. \tag{8}$$

We represent the integrand in the form

$$\int\limits_{y_0}^{y}\left(\frac{1}{y}-\frac{1}{y-1}\right)\mathrm{d}y=rt \tag{9}$$

and we get

$$\left(\ln|\,y\,|-\ln|\,y-1\,|\,\right)|_{y_0}^{y}=rt. \tag{10}$$

According to Newton-Leibniz formula,

$$\ln\left|\frac{y}{y-1}\right|-\ln\left|\frac{y_0}{y_0-1}\right|=rt. \tag{11}$$

As y∈(0;1), then

$$\left|\frac{y}{y-1}\right|=\frac{y}{1-y}. \tag{12}$$

Considering this condition and the fact, that the difference of the logarithms is equal to the logarithm of the fraction, we get:

$$\ln\frac{y(1-y_0)}{(1-y)y_0}=rt,\qquad \frac{y(1-y_0)}{(1-y)y_0}=e^{rt}, \tag{13}$$

$$\frac{1-y}{y}=\frac{1-y_0}{y_0e^{rt}},\qquad \frac{1}{y}-1=\frac{1-y_0}{y_0e^{rt}}, \tag{14}$$

$$\frac{1}{y}=\frac{1-y_0+y_0e^{rt}}{y_0e^{rt}}, \tag{15}$$

$$y=\frac{y_0e^{rt}}{1-y_0+y_0e^{rt}}. \tag{16}$$

Consider the behaviour of the solution Eq.(16) at infinity:

Fig. 1. Logistic function presented in *Wolfram Alpha* style.

$$\lim_{t\to+\infty}y(t)=\lim_{t\to+\infty}\left(\frac{y_0e^{rt}}{1+y_0(e^{rt}-1)}\right), \tag{17}$$

$$\lim_{t\to+\infty}y(t)=\lim_{t\to+\infty}\left(\frac{y_0}{\frac{1}{e^{rt}}+y_0(1-\frac{1}{e^{rt}})}\right)=\frac{y_0}{y_0}=1. \tag{18}$$

Using that substitutes

$$y=\frac{x}{k},\qquad y_0=\frac{x_0}{k} \tag{19}$$

in Eq.(16), we get

$$\frac{x}{k}=\frac{\frac{x_0}{k}e^{rt}}{1-\frac{x_0}{k}+\frac{x_0}{k}e^{rt}} \tag{20}$$

and, consequently, an exact solution of Eq.(2) is so called *logistic function*

$$x(t)=\frac{kx_0e^{rt}}{k+x_0(e^{rt}-1)}, \tag{21}$$

where x_0 represents the initial size of population. For function $x(t)=ky(t)$, according to Eq.(18),

$$\lim_{t\to+\infty}y(t)=1,\qquad \lim_{t\to+\infty}x(t)=k, \tag{22}$$

here k represents the capacity of the environment, as the maximum possible size of the population. This solution does not give periodic solutions or any deviations.

For drawing of the logistic function, comprehensive list of mathematical software includes *Mathcad* [11] and *Wolfram* [12]. For example, Fig. 1 represents the expression in *Wolfram* style. Fig. 2 represents the plot of logistic function expressed by Eq.(21).

Discrete analogue of the Verhulst equation. We consider the Verhulst equation

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ry(1-y). \tag{23}$$

Let's assume the discrete time scale: t=0,1,2, ... (time changes discretely). We denote by

$$y(0)=y_0,\quad y(1)=y_1,\quad y(2)=y_2,\quad \dots\ . \tag{24}$$

Generally,

$$y(t)=y_t,\quad y(t+1)=y_{t+1}, \tag{25}$$

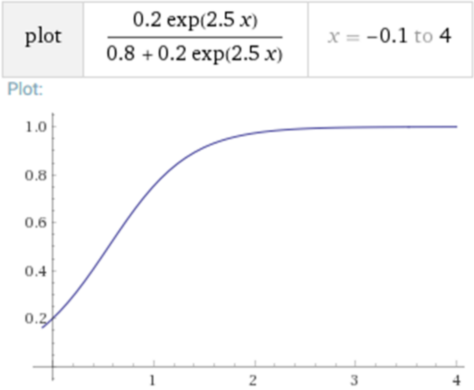

Fig. 2. Graph of logistic function expressed by Eq.(21): r=2.5; k=1.0; x_0=0.2.

where y_t represents the population size at year t. Since the derivative of the function is the limit of the ratio of the function increment to the argument increment, we can assume that the derivative is approximately equal to presented ratio:

$$\frac{\mathrm{d}y}{\mathrm{d}t} \approx \frac{\Delta y}{\Delta t}. \tag{26}$$

In our case

$$\Delta t = (t+1) - t = 1, \quad \Delta y = y_{t+1} - y_t. \tag{27}$$

For Verhulst equation Eq.(23), taking into account the approximation expressed by Eq.(26), we receive the equation

$$\frac{y_{t+1} - y_t}{1} = r y_t (1 - y_t), \tag{28}$$

$$y_{t+1} = (1+r) y_t - r y_t^2. \tag{29}$$

By transforming to another form

$$y_{t+1} = (1+r) y_t \left(1 - \frac{r}{1+r} y_t\right) \tag{30}$$

and after replacement

$$\frac{r}{1+r} y_t = x_t, \qquad y_t = \frac{1+r}{r} x_t, \tag{31}$$

as a result, we obtain the new one:

$$x_{t+1} = (1+r) x_t (1 - x_t). \tag{32}$$

Using r instead of (1+r), we obtain a discrete analogue of the Verhulst equation:

$$x_{t+1} = r x_t (1 - x_t). \tag{33}$$

We will study the properties of Eq.(33) at different values of the parameter r. Two-dimensional graphs presented in Figs. 3-4, 6-11 were prepared using *Cobweb* software [13].

Let's consider the construction path of the sequence (x_t). We use two functions y=x and y=$rx(1-x)$ presented in Fig. 3. The vertex of a parabola is a point with coordinates (0.5;r/4), x=0 and x=1 are zeros of the quadratic function.

First step. We take an arbitrary initial condition x_0. We draw a vertical line to the intersection with the parabola. On the y-axis we obtain the value x_1. Now we use a straight line y=x, we transfer the value x_1 to the x-axis.

Second step. At the point x_1, let's draw a vertical line to the intersection with the parabola. On the y-axis we obtain the value x_2. Now we use a straight line y=x, we transfer the value x_2 to the x-axis.

Third and following steps. This step will be realized using the same routine. Fig. 3 represents geometrical view of the sequence formation: $x_0, x_1, x_2, x_3, \ldots$, using the functions y=x and y=$rx(1-x)$, r=1.78.

If you do not follow the same path twice in the forward and backward directions, you may limit yourself to a broken line: a vertical line from the initial value x_0 to the intersection with the parabola, then the horizontal line to the intersection with the straight line y=x. Let's keep doing it again and again, the vertical line to the intersection with the parabola and the horizontal line to the intersection with the straight line y=x, etc, as shown in Fig. 4.

Let's consider the behaviour of the sequence (x_t) at different values of parameter r. For 0⩽r<3, three different characteristic types of behaviour could be established.

Let's consider Eq.(33) with parameters r=0.5, r=1.6, r=2.87 as examples for analysing. In case when r=0.50 (see Fig. 5), the sequence (x_t) converges to zero for any initial value x_0. In case when r=1.60 (see Fig. 6), the sequence (x_t)

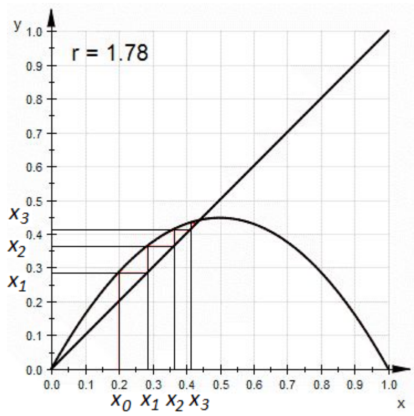

Fig. 3. Formation of sequence $x_0, x_1, x_2, x_3, \ldots$, using the functions y=x and y=rx(1-x), r=1.78, x_0=0.2.

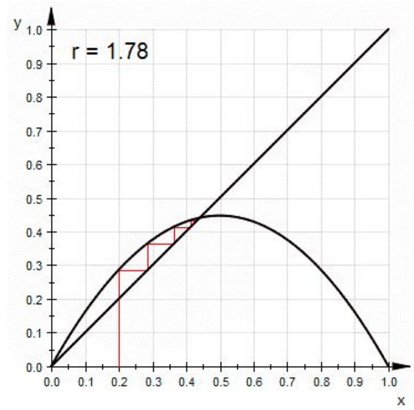

Fig. 4. Schematic representation of the sequence at r=1.78, x_0=0.2 (according to Fig. 3).

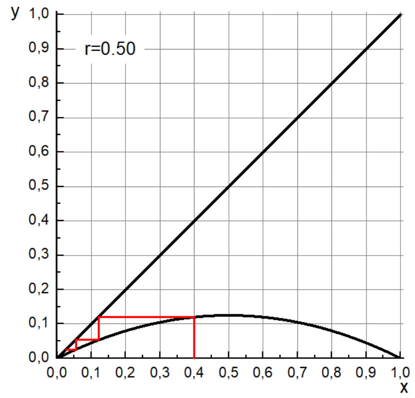

Fig. 5. Sequence (x_t) converges to x^*=0
for x_0=0.4 at r=0.50.

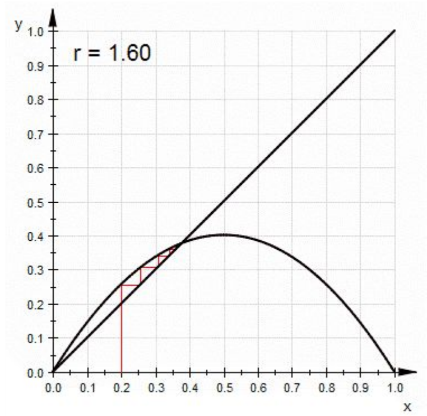

Fig. 6. Sequence (x_t) converges to x^*=0.375
for x_0=0.2 at r=1.60.

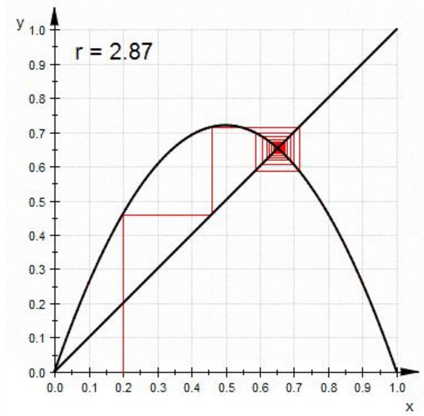

Fig. 7. Sequence (x_t) converges to $x^*\approx$0.65
for x_0=0.2 at r=2.87.

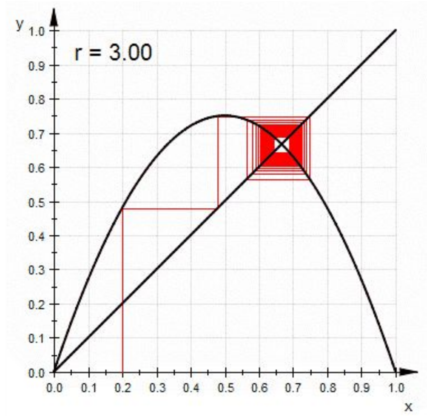

Fig. 8. Sequence (x_t) behaviour near the point x^*=2/3
at r=3.00.

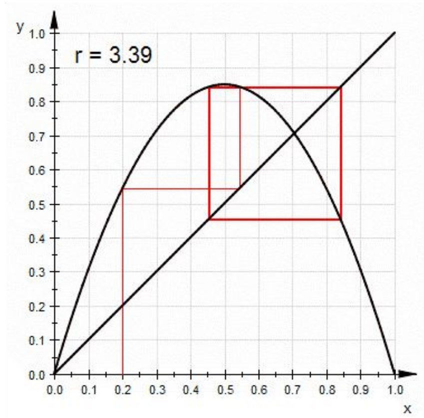

Fig. 9. Sequence (x_t). Oscillation between two values
$x\approx$0.84 and $x\approx$0.46 at r=3.39.

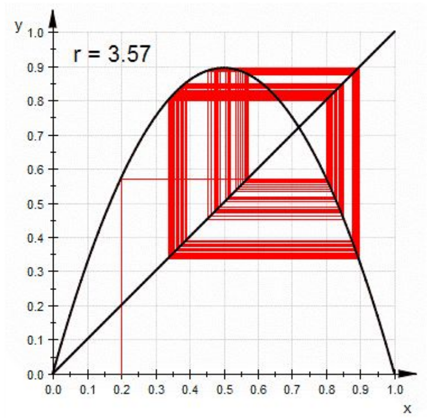

Fig. 10. Sequence (x_t) oscillates in chaotic manner
at r=3.57.

converges to x^*=0.375 quickly enough for any value x_0. In case when r=2.87 (see Fig. 7), the sequence (x_t) converges to the value $x^*\approx$0.65 quite slowly - not as fast as in case when r=1.60.

For 3$\leqslant r \leqslant$4, the behaviour of sequence (x_t) changes. Let's consider r=3.00, r=3.39, r=3.57, r=3.93 as examples for analysing. When r=3.00 (see Fig. 8), the sequence (x_t) approaches to the point x^*=2/3 very slowly. When r=3.39 (see Fig. 9), the sequence (x_t) oscillates between two values $x^*\approx$0.84 and $x^*\approx$0.46.

When r=3.57, convergence and oscillations are absent, and values of sequence (x_t) are distributed in chaotic manner with several groups of periodic frames - see Fig. 10. With increasing of r up to value 3.93, the same behaviour will be kept, and number of periodic frames increases - see Fig. 11.

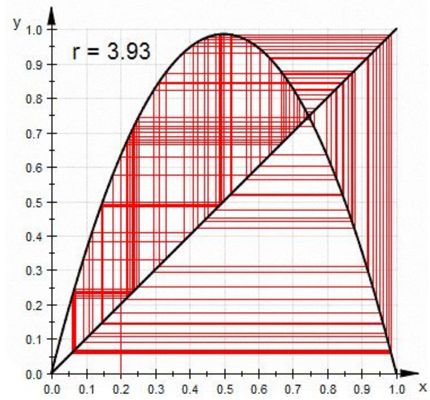

Fig. 11. Sequence (x_t) is distributed chaotic for any x_0 at r=3.93.

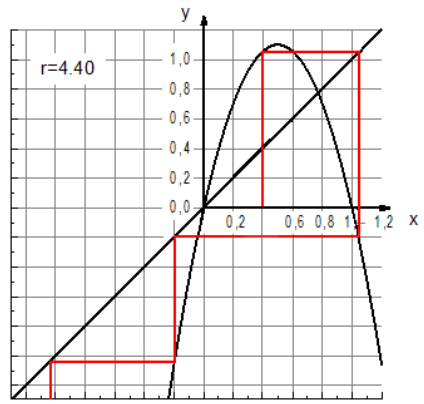

Fig. 12. Sequence (x_t) for any x_0 tends to $-\infty$ at r=4.40.

3. Attractors. Conditions for stability

Let's assume the transition of current system from one state to another. Let the transition be described by the equation

$$x_{t+1} = F(x_t) = rx_t(1 - x_t) \tag{34}$$

and the initial conditions of the system are chosen arbitrarily, then the final behaviour of the system is described by a point or a set of points.

A point or set of points that attract all nearest points to it is called an attractor. We will consider three kinds of attractors:

i) a fixed-point attractor;
ii) a limit cycle attractor or a periodic attractor;
iii) chaotic attractor or strange attractor.

According to the kind of attractors, the behaviour of a nonlinear system could be classified into following groups:

a) stable and converging to an equilibrium value;
b) oscillating in a stable limit cycle;
c) chaotic, but bounded;
d) unstable and exploding.

Using the equation Eq.(34) when $x_t \in [0;1]$, r must be treated as a variable parameter. We will consider $0 \leqslant r \leqslant 4$, since when $r>4$, sequence (x_t) tends to $-\infty$. This is an unstable and unlimited behaviour of the system: see Fig. 12 for sequence at r=4.40.

3.1. The fixed point attractor

In mathematics, a *fixed point* of a function is an element of the function's domain that is mapped to itself by the function. Accordingly, x^* is a fixed point of the function $F(x)$ if $F(x^*)=x^*$. This means

$$F^2(x^*) = F(F(x^*)) = F(x^*) = x^*, \tag{35}$$

$$F^n(x^*) = x^*, \qquad n \in N, \tag{36}$$

an important terminating consideration when recursively computing F.

Let's consider the equation

$$x = F(x). \tag{37}$$

Function F is a contracting map in a closed interval $I \in R$ if F meets two following conditions.
1. $F{:}I \to I$, where I is a closed interval. If $x \in I$, then $F(x) \in I$.
2. F is the contraction on this interval, if some $L \in (0;1)$ exists, such that inequality

$$\mid F(x) - F(x') \mid \leqslant L \mid x - x' \mid \tag{38}$$

is valid for any $x, x' \in I$.

Then, according to the principle of contracting map, the equation $x=F(x)$ has a unique solution $x^* \in I$, and for any initial condition $x_0 \in I$ sequence (x_t), t=0,1,2, ..., determined by the condition $x_{t+1}=F(x_t)$, converges to the value x^*.

Passing to the limit in Eq.(34), we get

$$x^* = rx^*(1 - x^*). \tag{39}$$

Therefore, we find the fixed point x^* by solving equation

$$x = F(x), \tag{40}$$

where

$$F(x) = rx(1 - x). \tag{41}$$

One solution of the equation

$$x = rx(1 - x). \tag{42}$$

is x=0. Then, if $x \neq 0$, $1=r \cdot (1-x)$, and another solution is

$$x = \frac{r-1}{r}. \tag{43}$$

It means that two fixed points will be determined as follows:

$$x_1^* = 0; \qquad x_2^* = \frac{r-1}{r}. \tag{44}$$

If $r \in (0;1)$, then $r-1<0$ and x_2^* does not belong to the interval [0;1].

Let's start to analyse the function F expressed from Eq.(41):

$$F(x) = rx - rx^2. \tag{45}$$

Table 2. Conditions of the stability at fixed point.

$\mid F'(x)\vert_{x=x^*} \mid$ <1	locally stable	attracting
$\mid F'(x)\vert_{x=x^*} \mid$ =1	neutral stable	non-attracting non-repulsive
$\mid F'(x)\vert_{x=x^*} \mid$ >1	unstable	repulsive

According to the condition that the point x^* is the limit of the sequence (x_t), it is necessary that F meets the second condition - see Ineq.(38). For $F(x)$, the first condition fulfilled at interval [0;1]. This means that if $x \in [0;1]$, then $F(x) \in [0;1]$. Second condition will be satisfied, if

$$\mid F'(x)|_{x=x^*} \mid < 1 \tag{46}$$

According to the condition $|F'(x)|_{x=x^*}|$ <1, if the tangent of the slope of the function graph at the point x^* lies in interval (-1;1), this fixed point is called as *locally stable*. If $|F'(x)|_{x=x^*}|$=1, the tangent to the graph of the function coincides with the line y=x or y=$-x$. When $|F'(x)|_{x=x^*}|$ =1, the point x^* is called as *neutral stable*: it ceases to be stable (attract sequence x_t), but not repulsive, i.e. is not unstable. In case if $F'(x)|_{x=x^*}$=0, point x^* is called as super-stable.

Conditions of the stability at fixed point are presented in Table 2.

Let's express the first derivative of F:

$$F'(x) = r - 2rx. \tag{47}$$

We calculate $F'(x)$ at points x_1^* and x_2^* using Eqs.(44):

$$F'(0) = r, \tag{48}$$

$$F'\Big(\frac{r-1}{r}\Big) = r - 2r \cdot \frac{r-1}{r} = r - 2r + 2 = 2 - r. \tag{49}$$

Note that $|r|$=r for $r \geqslant 0$ and

$$|2 - r| = |r - 2|. \tag{50}$$

Three possible cases for point x_2^* are presented below.
If $|r-2|$ <1, then -1<$r-2$<1, it means 1<r<3.
If $|r-2|$ =1, then $r-2$=-1 or $r-2$=1, it means r=1 or r=3.
If $|r-2|$ >1, then $r-2$<-1 or $r-2$>1, it means r<1 or r>3.
Table 3 represents the behaviour of the fixed points. Table 4 represents the dependence of the stability form at the fixed point on the value of the parameter r.

3.2. Periodic attractor

Table 4 represents the dependence of the stability form at the certain fixed points. Point x_2^*=$(r-1)/r$ becomes unstable, when r>3. According to that, the behaviour of the point x_2^* changes from attraction to repulsion when r=3. Fig. 9 represents the plot of two functions, y=x and y=$rx(1-x)$, when r=3.39. Two crossing points are x_1^*=0 and x_2^*≈0.7. The sequence (x_t) oscillates between two other points: x≈0.46 and x≈0.84.

Instead of one stable point, two new ones appear. It means that after a certain number of iterations the system begins to oscillate from one of these points to the other. These points can be found from equation x= $F^2(x)$.

Periodic attractor for x_{t+2}=$F^2(x_t)$**.** We know that

$$x_{t+2} = F(x_{t+1}) = F(F(x_t)) = F^2(x_t), \tag{51}$$

where

$$F(x_t) = rx_t(1 - x_t). \tag{52}$$

According to that,

$$F^2(x_t) = F(F(x_t)), \tag{53}$$

$$F^2(x_t) = F(rx_t(1 - x_t)), \tag{54}$$

$$F^2(x_t) = r^2x_t(1 - x_t)(1 - rx_t(1 - x_t)), \tag{55}$$

$$F^2(x_t) = r^2x_t(1 - x_t)(1 - rx_t + rx_t^2), \tag{56}$$

$$F^2(x_t) = r^2x_t(1 - rx_t - x_t + 2rx_t^2 - rx_t^3). \tag{57}$$

We denote by $x^{*(2)}$ the fixed points of equation

$$x = F^2(x). \tag{58}$$

To find them, let's solve equation

$$x = r^2x(1 - rx - x + 2rx^2 - rx^3). \tag{59}$$

Similar as in previous case,

$$x_1^{*(2)} = 0. \tag{60}$$

Table 3. Behaviour of the fixed points.

Fixed point	Locally stable (attracting) $\mid F'(x)\vert_{x=x^*} \mid$ <1	Neutral stable $\mid F'(x)\vert_{x=x^*} \mid$ =1	Unstable $\mid F'(x)\vert_{x=x^*} \mid$ >1
x_1^*=0	0⩽r<1	r=1	r>1
x_2^*=$(r-1)/r$	1<r<3	r=1 or r=3	r<1 or r>3

Table 4. Dependence of the form of stability at a fixed point on the value of the parameter r

	r∈(0;1)	r=1	r∈(1;3)	r=3	r∈(3;4]
x_1^*=0	Locally stable	Neutral stable	Unstable	Unstable	Unstable
x_2^*=$(r-1)/r$	x_2^*<0	x_2^*=0=x_1^*	Locally stable	Neutral stable	Unstable

Table 5. Horner's scheme.

	r^3	$-2r^3$	r^3+r^2	$1-r^2$
		+	+	+
$\frac{r-1}{r}$	$\downarrow$	r^3-r^2	$-(r-1)(r^2+r)$	r^2-1
		=	=	=
	r^3	$-r^3-r^2$	r^2+r	0

We divide both sides of the Eq.(59) by $x{\neq}0$.

$$1 = r^2(1 - rx - x + 2rx^2 - rx^3). \tag{61}$$

We transform it to the form of cubic equation:

$$r^3x^3 - 2r^3x^2 + r^2(r+1)x + 1 - r^2 = 0 \tag{62}$$

and use the Horner's scheme. It is known that

$$x_2^* = \frac{r-1}{r} \tag{63}$$

is the root of Eq.(62). We use Horner's scheme presented in Table 5. We can find the remaining two roots solving the equation

$$r^3x^2 - r^2(1+r)x + r(r+1) = 0. \tag{64}$$

As $r{\neq}0$, we divide the equation by r^2:

$$rx^2 - (1+r)x + \frac{r+1}{r} = 0. \tag{65}$$

Let's calculate the discriminant D:

$$D = (1+r)^2 - 4(r+1) = r^2 - 2r - 3 \tag{66}$$

and express the solution in form:

$$x_{3,4}^{*(2)} = \frac{1 + r \pm \sqrt{r^2 - 2r - 3}}{2r}. \tag{67}$$

Depending on the sign of the discriminant, we need to consider three different cases.

First case. $D{>}0$, if $r{<}{-1}$ or $r{>}3$. In this case for $r{>}3$ we get two additional solutions:

$$x_{3,4}^{*(2)} = \frac{1 + r \pm \sqrt{r^2 - 2r - 3}}{2r}. \tag{68}$$

Second case. $D{=}0$, if $r{=}{-1}$ or $r{=}3$. At $r{=}3$, we obtain a solution:

$$x_3^{*(2)} = x_4^{*(2)} = \frac{1+r}{2r} = \frac{1+3}{2\cdot 3} = \frac{2}{3}, \tag{69}$$

which coincides with the solution

$$x_2^{*(2)} = \frac{r-1}{r} = \frac{2}{3}. \tag{70}$$

Third case. $D{<}0$, if $-1{<}r{<}3$. There are no additional solutions.

We obtain the following dependence of the amount of limit points on the parameter r as presented in Table 6. To check which of these points are attractive, you need to count the first derivative of $F^2(x)$ on x at these points and make sure that the inequality Ineq.(71) is satisfied:

$$\left| \left. \frac{\mathrm{d}F^2(x)}{\mathrm{d}x} \right|_{x=x^*} \right| < 1. \tag{71}$$

Example. For $r{=}3.4$ we obtain four fixed points:

$$x_1^{*(2)} = 0; \tag{72}$$

$$x_2^{*(2)} = \frac{3.4-1}{3.4} = \frac{24}{34} = \frac{12}{17} \approx 0.706; \tag{73}$$

$$x_{3,4}^{*(2)} = \frac{3.4 + 1 \pm \sqrt{3.4^2 - 2\cdot 3.4 - 3}}{2\cdot 3.4}; \tag{74}$$

$$x_3^{*(2)} = \frac{4.4 - \sqrt{1.76}}{6.8} \approx 0.452; \tag{75}$$

$$x_4^{*(2)} = \frac{4.4 + \sqrt{1.76}}{6.8} \approx 0.842. \tag{76}$$

For $F^2(x)$ (see Eq.(57)), let's calculate the first derivative with respect to x:

$$F^2(x) = r^2(x - rx^2 - x^2 + 2rx^3 - rx^4); \tag{77}$$

$$\frac{\mathrm{d}F^2(x)}{\mathrm{d}x} = r^2(1 - 2rx - 2x + 6rx^2 - 4rx^3). \tag{78}$$

Now we will establish the form of stability at fixed points. The point $x_1^{*(2)}{=}0$ is an unstable fixed point because

$$\left. \frac{\mathrm{d}F^2(x)}{\mathrm{d}x} \right|_{x=x_1^{*(2)}} = r^2 = 3.4^2 > 1. \tag{79}$$

Table 6. Dependence of the amount of limit points on the parameter r.

$0{<}r{<}3$, two points	$r{=}3$, two points	$r{>}3$, four points
$x_1^{*(2)}{=}0$ $x_2^{*(2)}{=}\frac{r-1}{r}$	$x_1^{*(2)}{=}0$ $x_2^{*(2)}{=}x_3^{*(2)}{=}x_4^{*(2)}{=}\frac{2}{3}$	$x_1^{*(2)}{=}0$ $x_2^{*(2)}{=}\frac{r-1}{r}$ $x_{3,4}^{*(2)}{=}\frac{1+r\pm\sqrt{r^2-2r-3}}{2r}$

The point $x_2^{*(2)}\approx$0.706 is an unstable fixed point because

$$\left.\frac{\mathrm{d}F^2(x)}{\mathrm{d}x}\right|_{x=x_2^{*(2)}} \approx 3.4^2(1-2\cdot 0.706-2\cdot 3.4\cdot 0.706+6\cdot 3.4\cdot 0.706^2-4\cdot 3.4\cdot 0.706^3)\approx 1.96>1. \quad (80)$$

The point $x_3^{*(2)}\approx$0.452 is locally stable fixed point because

$$\left.\frac{\mathrm{d}F^2(x)}{\mathrm{d}x}\right|_{x=x_3^{*(2)}} \approx 3.4^2(1-2\cdot 0.452-2\cdot 3.4\cdot 0.452+6\cdot 3.4\cdot 0.452^2-4\cdot 3.4\cdot 0.452^3)\approx -0.759. \quad (81)$$

The point $x_4^{*(2)}\approx$0.842 is locally stable fixed point because

$$\left.\frac{\mathrm{d}F^2(x)}{\mathrm{d}x}\right|_{x=x_4^{*(2)}} \approx 3.4^2(1-2\cdot 0.842-2\cdot 3.4\cdot 0.842+6\cdot 3.4\cdot 0.842^2-4\cdot 3.4\cdot 0.842^3)\approx -0.754. \quad (82)$$

In case of r=3.4, existence of two stable points is established. Fig. 13 represents the plots of functions y=x and y=$F^2(x)$ (see Eq.(57)). Tangent of the slope of the graph at points $x_3^{*(2)}\approx$0.452 and $x_4^{*(2)}\approx$0.842 lies in interval ($-$1;0).

We consider the case, when r=3.51 - see Fig. 14. We can see that all four crossing points of the functions y=x and y=$F^2(x)$ are unstable. For the value of the parameter r=3.51, points $x_3^{*(2)}$ and $x_4^{*(2)}$ cease to be stable and both points generate two new points (each). This phenomenon is called period doubling. Fig. 15 represents the functions y=x and y=$F^4(x)$, where four new crossing points $x_5^{*(4)}$, $x_6^{*(4)}$, $x_7^{*(4)}$ and $x_8^{*(4)}$ are appearing.

When r increases further, four stable points cease to be stable and generate eight new points. The points at which the solution doubles are called as *bifurcation points*. The bifurcation process continues, generating 16, 32, 64, ... stable points. These points can be found, and their stability is determined in the same way as in previous case.

When a stable point ceases to be stable, it no longer attracts points. However, if the value of the unstable fixed point is given as the initial condition of the system, then the stable fixed points do not attract these points, the system remains at these points. The point x=1 is not fixed, so as $F(1)$=0, then the initial condition x_0=1 generates the sequence x_1=0, x_2=0, x_3=0, ..., x_t=0, ...

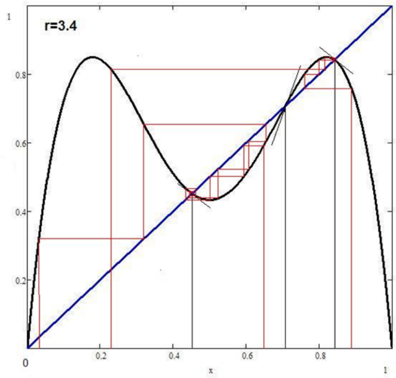

Fig. 13. y=x and y=$F^2(x)$ at r=3.4. Four crossing points, but only two of them are stable for any x_0.

Periodic attractor for x_{t+4}=$F^4(x_t)$. According to Eq.(55) we know that

$$F^2(x_t)=r^2x_t(1-x_t)(1-rx_t(1-x_t)). \quad (83)$$

Denote by

$$B(x_t)=1-rx_t(1-x_t), \quad (84)$$

then

$$F^2(x_t)=r^2x_t(1-x_t)B(x_t). \quad (85)$$

Let's calculate F^3:

$$F^3(x_t)=F(F^2(x_t)), \quad (86)$$

$$F^3(x_t)=rF^2(x_t)(1-F^2(x_t)), \quad (87)$$

$$F^3(x_t)=r^3x_t(1-x_t)B(x_t)(1-F^2(x_t)). \quad (88)$$

Let's denote by

$$C(x_t)=(1-x_t)(1-F^2(x_t)), \quad (89)$$

then

$$F^3(x_t)=r^3x_tB(x_t)C(x_t). \quad (90)$$

Finally, let's calculate F^4:

$$F^4(x_t)=F(F^3(x_t)), \quad (91)$$

$$F^4(x_t)=rF^3(x_t)(1-F^3(x_t)), \quad (92)$$

$$F^4(x_t)=r^4x_tB(x_t)C(x_t)(1-r^3x_tB(x_t)C(x_t)), \quad (93)$$

where $F^4(t_x)$ is the sixteenth-order polynomial with respect to x_t. We denote by $x^{*(4)}$ the fixed points of equation

$$x=F^4(x). \quad (94)$$

These solutions could be obtained by means of *Mathcad* software [11] using routine:

$$F^4(x)=x \quad solve, x\rightarrow \quad (95)$$

There are sixteen solutions of Eq.(94): eight solutions in real form and eight solutions in complex form as presented in Table 7. Only real solutions are considered here. Complex solutions are out of our interest. Among the set of solutions, $x_1^{*(4)}$, $x_2^{*(4)}$, $x_3^{*(4)}$, $x_4^{*(4)}$ correspond to the $x_1^{*(2)}$, $x_2^{*(2)}$, $x_3^{*(2)}$, $x_4^{*(2)}$ respectively as solutions of equation x=$F^2(x)$. Also, new solutions $x_5^{*(4)}$, $x_6^{*(4)}$, $x_7^{*(4)}$, $x_8^{*(4)}$ appear. Fig. 15 represents the plots of functions y=x and y=$F^4(x)$ (Eq.(93)), where eight crossing points are indicated.

Table 7. Solutions of Eq.(94) at r=3.51 obtained using *Mathcad* [11].

Real solutions		Corresponds to the fixed point of F^2	Equation	Complex solutions	
n	$x_n^{*(4)}$			k	
8	0.87734182			1	0.98584709 − 0.00694482 i
4	0.85849118	$x_4^{*(4)} \to x_4^{*(2)}$	Eq.(67)	2	0.98584709 + 0.00694482 i
7	0.82501893			3	0.50562789 + 0.17578337 i
2	0.71509972	$x_2^{*(4)} \to x_2^{*(2)}$	Eq.(63)	4	0.50562789 − 0.17578337 i
6	0.50671306			5	0.16598412 + 0.07496759 i
3	0.42640911	$x_3^{*(4)} \to x_3^{*(2)}$	Eq.(67)	6	0.16598412 − 0.07496759 i
5	0.37772216			7	0.04914291 + 0.02368632 i
1	0	$x_1^{*(4)} \to x_1^{*(2)}$	Eq.(60)	8	0.04914291 − 0.02368632 i

3.3. Population behaviour and parameter r

If $r \in (0;1)$, the population will die out, regardless of the initial conditions.

$$x_1^* = 0. \tag{96}$$

If $r \in (1;2)$, the population size will quickly reach the stationary value

$$x_2^* = \frac{r-1}{r}, \tag{97}$$

regardless of the initial conditions.

If $r \in (2;3)$, the population size will also come to the same stationary value x_2^*, but will initially oscillate around it.

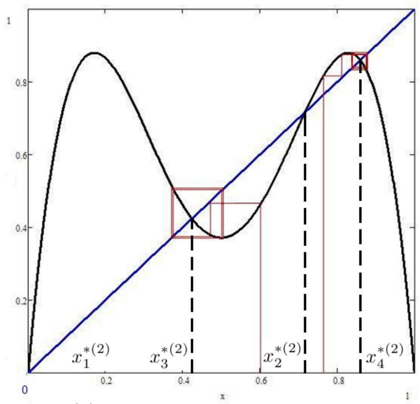

Fig. 14. y=x and y=$F^2(x)$ at r=3.51. Four crossing points, all of them are unstable. Fixed points $x_n^{*(2)}$, n=1,2,3,4, are indicated by dashed line.

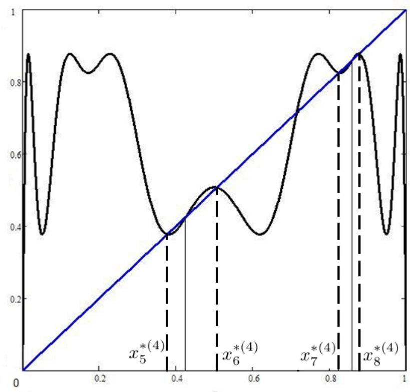

Fig. 15. y=x and y=$F^4(x)$ at r=3.51. Eight crossing points. Fixed points $x_n^{*(4)}$, n=5,6,7,8, are indicated by dashed line.

If $r \in (3;1+\sqrt{6})$, where $1+\sqrt{6}$=3.4495≈3.45, the population will fluctuate infinitely between two values [8],

$$x_{3,4}^{*(2)} = \frac{1+r \pm \sqrt{r^2-2r-3}}{2r} \tag{98}$$

and their value does not depend on x_0.

If $r \in (1+\sqrt{6};\ 3.54)$, then the population size will fluctuate between four values.

If r>3.54, then the population size will fluctuate between 8 values, then 16, 32, etc.

Table 8 represents the dependence of fixed points x^* on r for bifurcational diagram. Bifurcation diagram demonstrates current attractor points for r values - see Fig. 16. The length of the interval at which the oscillations occur between the same number of values decreases as r increases. The ratio between the two system interval lengths tends to the first Feigenbaum constant b=4.669201609…[3]. Such behavior is a typical example of a period doubling bifurcation cascade.

If r≈3.57, chaotic behaviour begins, and the doubling cascade ends. Fluctuations are no longer observed. Slight changes in the initial conditions lead to incomparable differ-

Table 8. Fixed point values for bifurcational diagram.

r	x_2^*			
1.0	0			
1.5	0.33			
2.0	0.50			
2.5	0.60			
3.0	0.67			
r	$x_3^{*(2)}$	$x_4^{*(2)}$		
3.0	0.67	0.67		
3.1	0.56	0.76		
3.2	0.51	0.79		
3.3	0.48	0.82		
3.4	0.45	0.84		
3.45	0.44	0.85		
r	$x_5^{*(4)}$	$x_6^{*(4)}$	$x_7^{*(4)}$	$x_8^{*(4)}$
3.45	0.439	0.446	0.850	0.852
3.47	0.403	0.479	0.835	0.866
3.49	0.389	0.495	0.829	0.872
3.51	0.378	0.507	0.825	0.877
3.52	0.373	0.512	0.823	0.879
3.54	0.365	0.522	0.820	0.883

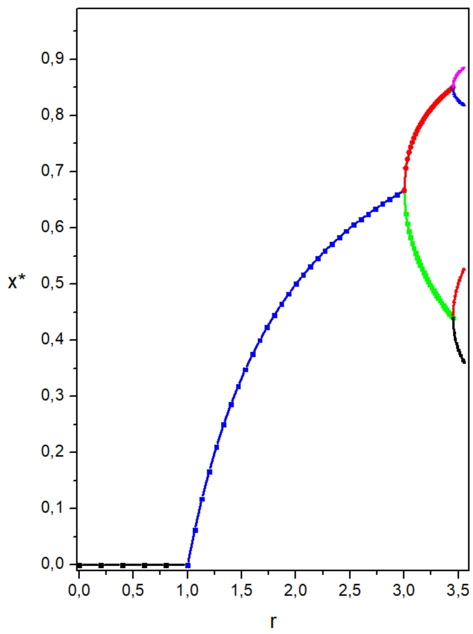

Fig. 16. Dependence of the attractor points x^* on parameter r. Bifurcation diagram.

ences in the future behaviour of the system in time, which is the main characteristic of chaotic behaviour.

For r>4, the display values leave the interval [0;1] and diverge under any initial conditions.

Conclusions

The discrete analogue of the Verhulst equation is interesting due to the following circumstance: for sequence Eq.(33) at different values of the parameter r, a set of different attractors (fixed point attractor, periodic attractor, chaotic attractor) could be obtained.

Methodologically, modelling tasks constructed using the Verhulst equation enable to understand the chaotic behaviour in real complicated forms of global complexity.

It is necessary to point out that chaotic behaviour of the model system corresponding to the real system depends on the method precision. Sensitivity of the model on initial conditions requires the detailed analysis of the stationary as well as dynamic behaviour.

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