# On the polynomials of optimal shape generating maximum number of period annuli

Yelena Kozmina [a]
Department of Natural Sciences and Computer Technologies
Information Systems Management Institute
Ludzas iela 91, LV-1003, Riga, Latvia

*Received 1 June 2012, accepted 11 November 2012*

**Abstract.** The nonlinear differential equation $x^{''}+g(x)=0$ is being considered, where $g(x)$ is a polynomial that allows the equation to have multiple period annuli. It is shown how the respective optimal polynomials can be constructed in case, when a primitive of the function $g(x)$ is a polynomial of an arbitrary selected even degree.

**Citations:** Yelena Kozmina. On the polynomials of optimal shape generating maximum number of period annuli – *Innovative Infotechnologies for Science, Business and Education,* ISSN 2029-1035 – **2(13)** 2012 – Pp. 13-19.

**Keywords:** nonlinear; differential; equation; periodic; annuli.
**Short title:** On the polynomials.

## Introduction

Lets consider the second order nonlinear autonomous differential equation

$$x^{''}+g(x)=0 \qquad (1)$$

where $g(x)$ is a polynomial of odd degree with simple zeros only and the highest degree of polynomial with a negative coefficient. Zero $z$ is called simple if $g(z)=0$ and $g'(z)\neq 0$. In these points function $G(x)$ which is a primitive of the function $g(x)$ has an alternate local maxima and minima points.

$$G(x)=\int_0^x g(s)ds \qquad (2)$$

Note that a critical point is a center, if it has a punctured neighborhood covered with nontrivial cycles. Period annuli looks like a domain filled with closed trajectories. Every connected region on a phase plane covered with nontrivial concentric closed curves is called a period annulus. We will call a period annulus associated with a central region by a trivial period annulus.

Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center. Period annuli enclosing several (more than one) critical points are called nontrivial period annuli. According to previous notes in Ref. [1-3], we are interested mostly in presence of nontrivial period annuli.

## 1. Methods

*Definition 1.* Points $x_i$ and $x_j$ of local maxima of the function $G(x)$ are called non-neighbouring, if there exists at least one point of local maxima of the function $G(x)$ in the interval $(x_i,x_j)$.

*Definition 2.* Two non-neighboring points of maxima $x_i$ and $x_j$ of $G(x)$ will be called a regular pair if $G(x)<\min(G(x_i),G(x_j))$ at any other point of maximum lying in the interval $(x_i,x_j)$.

*The theorem of existence of periodic annuli* is formulated according to Ref. [1]. If $g(x)$ is a polynomial of odd degree with simple zeros only, $G(x)$ is a primitive of the function $g(x)$, two points of maxima $x_i$ and $x_j$ of the function $G(x)$ form a regular pair, then the equation $x^{''}+g(x)=0$ has a nontrivial period annulus associated with the pair $x_i,x_j$.

*The theorem of the maximum number of periodic annuli* is formulated according to Ref. [2]. If $g(x)$ is a polynomial of odd degree with simple zeros only and the highest degree of polynomial with a negative coefficient, $G(x)$ is a primitive of the function $g(x)$, $n$ is the number of points of local maxima of the function $G(x)$, then the maximal possible number of regular pairs for $G(x)$ is equal to $(n-2)$.

*Proof.* Let $n=3$. The following $G(x)$ combinations are possible at three points of maxima as presented in Fig. 1.

A regular pair exists in the unique case presented in Fig. 1b, therefore, the maximal number of regular pairs is equal to one.

[a]email: *jelena.kozmina@isma.lv*

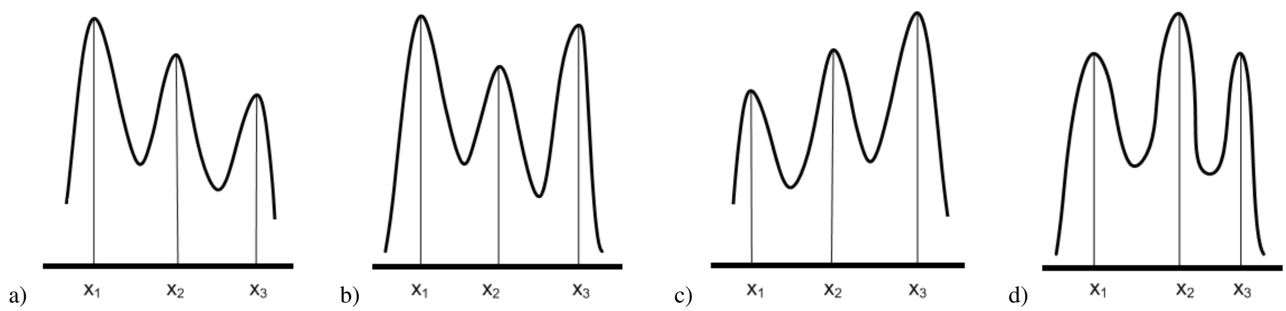


Fig. 1. $G(x)$ combinations at three maxima points.
a) $G(x_1) \geq G(x_2) \geq G(x_3)$, b) $G(x_2) < G(x_1)$, $G(x_2) < G(x_3)$,
c) $G(x_1) \leq G(x_2) \leq G(x_3)$, d) $G(x_2) \geq G(x_1)$, $G(x_2) \geq G(x_3)$.

Suppose that for any sequence of $k$ consecutive points of maxima of $G(x)(k \geq 3)$ the maximal number of regular pairs is equal to $(k-2)$. Without the loss of generality let us add to the right side the additional maximum point of the function $G(x)$. We get a sequence of $(k+1)$ consecutive points of maximum of the function $G(x) : x_1, x_2, \ldots, x_k, x_{k+1}$, when $x_1 < x_2 < \ldots < x_k < x_{k+1}$.

Let us prove that the maximal number of regular pairs is equal to $(k-1)$.

Let $x_p$, when $2 \leq p \leq k$, be one of the maxima points of $G(x)$ in which the function $G(x)$ has the largest value. Now cut the segment $[x_1, x_{k+1}]$ into two parts: $[x_1, x_p]$ and $[x_p, x_{k+1}]$.

Number of the points of maxima on the segment $[x_1, x_p]$ is equal to $p$ and the maximal possible number of regular pairs is equal to $(p-2)$. Number of the points of maxima on the segment $[x_p, x_{k+1}]$ is equal to

$$k + 1 - (p - 1) = k - p + 2 \leq k \tag{3}$$

and the maximal possible number of regular pairs is equal to $(k-p)$. Number of the maxima points on the segment $[x_1, x_{k+1}]$ is equal to $(k+1)$. Maximal possible number of regular pairs is equal to

$(p-2) + (k-p) + 1 = k - 1$, if $G(x_p) < G(x_1)$ and $G(x_p) < G(x_{k+1})$, and (see Fig. 2)

$(p-2) + (k-p) = k - 2$, if $G(x_p) \geq G(x_1)$ or $G(x_p) \geq G(x_{k+1})$, and

the additional regular pair does not appear. In a particular case (see Fig. 3) $G(x_1) > G(x_k)$ and

$$G(x_2) < G(x_3) < ... < G(x_k) < G(x_{k+1}), \tag{4}$$

the following regular pairs emerge, namely $x_1$ and $x_3$, $x_1$ and $x_4$, ... , $x_1$ and $x_k$, $x_1$ and $x_{k+1}$, in total $(k-1)$ pairs.

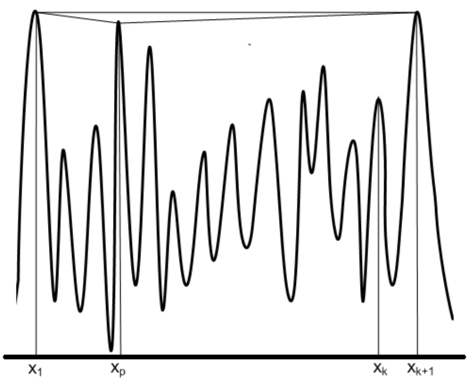


Fig. 2. Case of $G(x_p) < G(x_1)$ and $G(x_p) < G(x_{k+1})$, where one additional regular pair appears.

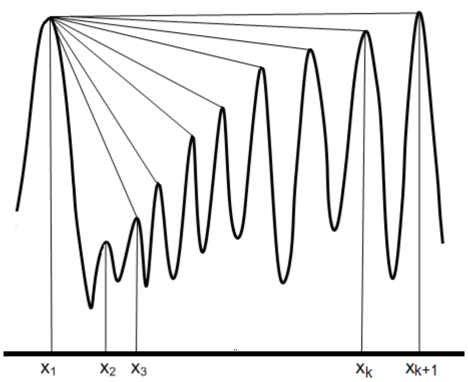


Fig. 3. Particular case with exactly $(k-1)$ regular pairs.

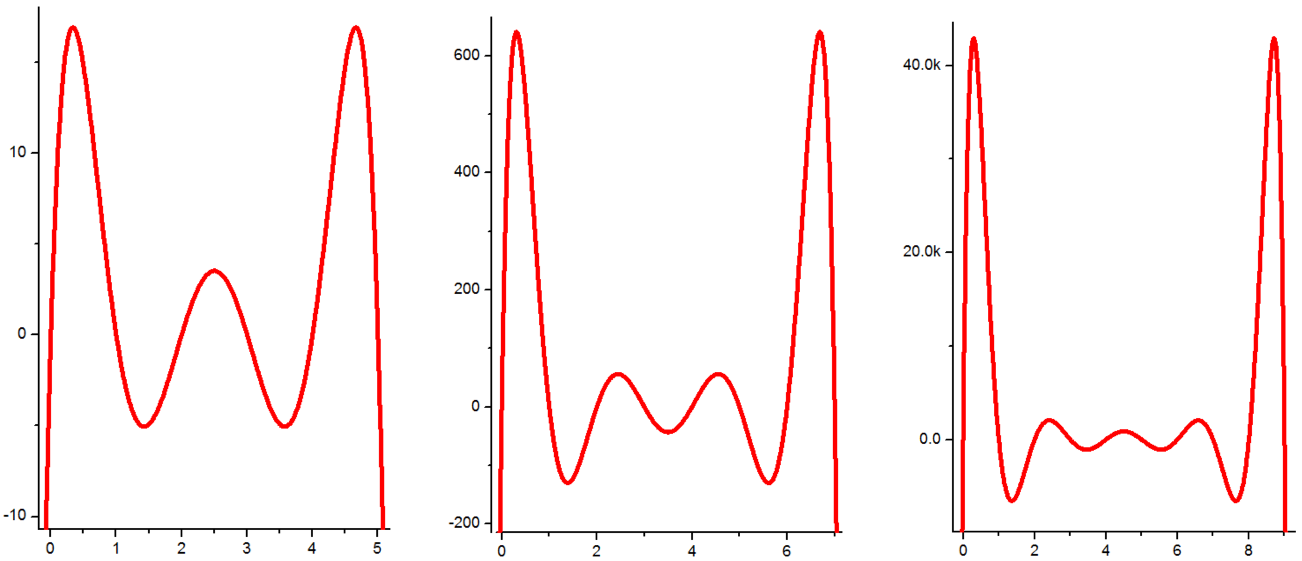


Fig. 4. Polynomial $P_{2n}(x)$ in cases, when $n = 3$ (left), $n = 4$ (center), $n = 5$ (right).

*Proposition 1.* Graph of the polynomial

$$P_{2n}(x) = -x(x-1)(x-2)\dots(x-(2n-1)) \tag{5}$$

is symmetrical relative to the straight line $x = n - \frac{1}{2}$ (see Fig. 4).

*Proof.* Lets provide the parallel shift of coordinate system

$$\begin{cases} x - \left(n - \dfrac{1}{2}\right) = x', \\ y = y' \end{cases} \tag{6}$$

to the point $O'(n - \frac{1}{2}, 0)$. In that case the polynomial $P_{2n}(x)$ is transformed to new one as follows:

$$P_{2n}(x') = -\left(x'^2 - \left(\frac{1}{2}\right)^2\right)\left(x'^2 - \left(\frac{3}{2}\right)^2\right)\dots\left(x'^2 - \left(n - \frac{1}{2}\right)^2\right). \tag{7}$$

$P_{2n}(x')$ is an even function, i.e., a graph is symmetrical to axis $O'Y'$.

*Proposition 2.* The polynomial

$$P_{2n}(x) = -x(x-1)(x-2)\dots(x-(2n-1)) \tag{8}$$

has $n$ maxima points $x_1, x_2, \dots x_n$, where $x_i \in (2i-2; 2i-1), i = 1, 2, \dots, n$, and

$$P_{2n}(x_1) = P_{2n}(x_n) > P_{2n}(x_2) = P_{2n}(x_{n-1}) > \dots > P_{2n}(x_{\frac{n}{2}}) = P_{2n}(x_{\frac{(n+2)}{2}}), \tag{9}$$

if $n$ is even,

$$P_{2n}(x_1) = P_{2n}(x_n) > P_{2n}(x_2) = P_{2n}(x_{n-1}) > \dots > P_{2n}(x_{\frac{(n+1)}{2}}), \tag{10}$$

if $n$ is odd.

## 2. Proof for $P_{2n}(x)$ where $n = 2$.

*Proof.* Let us consider polynomial $P_{2n}(x)$ where $n = 2$ and polinomial $P_4'(x)$ as the derivative of $P_4(x)$:

$$P_4(x) = -x(x-1)(x-2)(x-3), \tag{11}$$

$$P_4'(x) = -(4x^3 - 18x^2 + 22x - 6) = -2(2x-3)(x^2 - 3x + 1). \tag{12}$$

The polynomial $P_4(x)$ has two maxima points $x_1$ and $x_2$ related using expression: $P_4(x_1) = P_4(x_2)$:

$$x_1 = \frac{(3 - \sqrt{5})}{2} \in (0; 1); \qquad x_2 = \frac{(3 + \sqrt{5})}{2} \in (2; 3), \tag{13}$$

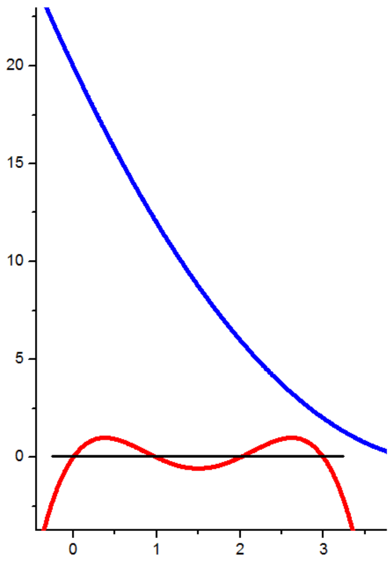


Fig. 5. $P_4(x) = -x(x-1)(x-2)(x-3)$ at bottom and $Q_2(x) = (x-4)(x-5)$ at top.

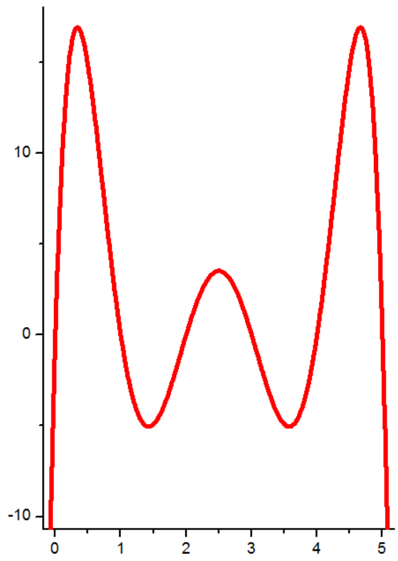


Fig. 6. Polynomial $P_6(x) = P_4(x) \cdot Q_2(x)$ $P_6(x) = -x(x-1)(x-2)(x-3)(x-4)(x-5)$

Polynomial $P_6(x)$ (see Fig. 5, Fig. 6) will be formed using the polynomial $P_4(x)$ and quadratic trinomial $Q_2(x)$:

$$Q_2(x) = (x-4)(x-5): \tag{14}$$

$$P_6(x) = -x(x-1)(x-2)(x-3)(x-4)(x-5) = P_4(x) \cdot Q_2(x). \tag{15}$$

Function $P_6(x)$ has two maxima points on segment $[0;3]$ belonging to the different intervals: $x_1^{'} \in (0;1)$ and $x_2^{'} \in (2;3)$, and derivatives of $P_6(x)$ at fixed points could be expressed as follow:

$$P_6(x_1^{'}) = \max_{x\in(0;1)} P_6(x) \geq P_6(x_1) = P_4(x_1) \cdot Q_2(x_1) > P_4(x_1) \cdot Q_2(1) = 12 \cdot P_4(x_1), \tag{16}$$

$$P_6(x_2^{'}) = P_4(x_2^{'}) \cdot Q_2(x_2^{'}) \leq \max_{x\in(2;3)} P_4(x) \cdot Q_2(x_2^{'}) < P_4(x_2) \cdot Q_2(2) = 6 \cdot P_4(x_2). \tag{17}$$

We obtain

$$P_6(x_1^{'}) > 12 \cdot P_4(x_1) > 6 \cdot P_4(x_2) > P_6(x_2^{'}) \tag{18}$$

On every segment between zeros of the polynomial $P_6(x)$ this polynomial has only one extremum point, because $P_6^{'}(x)$ is a polynomial of the $5^{th}$ degree, and can not have more than 5 zeros.

By symmetry of the graph of the polynomial $P_6(x)$ with respect to the line $= \frac{5}{2}$ , the second point of maxima is $x_2^{'} = \frac{5}{2}$, the third point of maxima is $x_3^{'} \in (4;5)$, and $P_6(x_3^{'}) = P_6(x_1^{'})$.

## 3. Proof for $P_{2n}(x)$ where $n = 3$.

Let us consider polynomial $P_{2n}(x)$ where $n = 3$. Polynomial $P_8(x)$ (see Fig. 7, Fig. 8) will be formed using the polynomial $P_6(x)$ and quadratic trinomial $Q_2(x)$:

$$P_6(x) = -x(x-1)(x-2)(x-3)(x-4)(x-5), \tag{19}$$

$$Q_2(x) = (x-6)(x-7), \tag{20}$$

$$P_8(x) = -x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7) = P_6(x) \cdot Q_2(x). \tag{21}$$

On segment $[0;3,5]$ the function $P_8(x)$ has two points of maxima $x_1^{''} \in (0;1)$ and $x_2^{''} \in (2;3)$, and

$$P_8(x_1^{''}) = \max_{x\in(0;1)} P_8(x) \geq P_8(x_1^{'}) = P_6(x_1^{'}) \cdot Q_2(x_1^{'}) > P_6(x_1^{'}) \cdot Q_2(1) = 30 \cdot P_6(x_1^{'}), \tag{22}$$

$$P_8(x_2^{''}) = P_6(x_2^{''}) \cdot Q_2(x_2^{''}) \leq \max_{x\in(2;3)} P_6(x) \cdot Q_2(x_2^{''}) < P_6(x_2^{'}) \cdot Q_2(2) = 20 \cdot P_6(x_2^{'}). \tag{23}$$

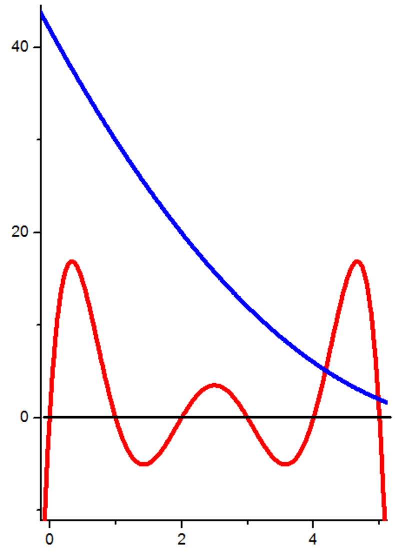


Fig. 7. Polynomial $P_6(x)$ at bottom
$P_6(x) = -x(x-1)(x-2)(x-3)(x-4)(x-5)$
and quadratic trinomial $Q_2(x) = (x-6)(x-7)$ at top.

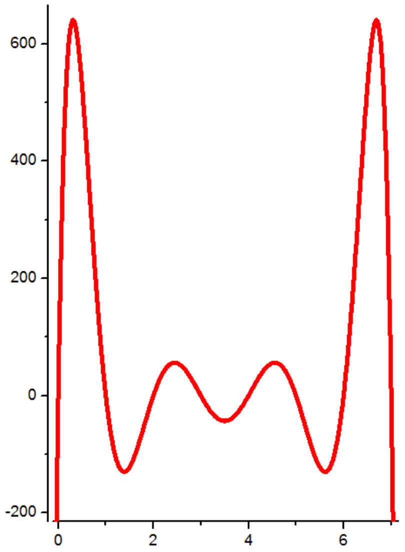


Fig. 8. Polynomial $P_8(x) = P_6(x) \cdot Q_2(x)$
$P_8(x) = -x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)$

We obtain

$$P_8(x_1^{''}) > 30 \cdot P_6(x_1^{'}) > 20 \cdot P_6(x_2^{'}) > P_8(x_2^{''}). \tag{24}$$

Polynomial $P_8(x)$ has two more points of maxima: $x_3^{''} \in (4;5)$ and $x_4^{''} \in (6;7)$.
By symmetry of the graph of the polynomial with respect to the line $x = \frac{7}{2}$, following expressions take place:

$$P_8(x_3^{''}) = P_8(x_2^{''}) \tag{25}$$

$$P_8(x_4^{''}) = P_8(x_1^{''}) \tag{26}$$

.

## 4. Proof for $P_{2n}(x)$ where $n = k+1$.

Suppose that the proposition 2 is true, if $n = k$, i.e., $P_{2k}(x)$ has $k$ points of maxima $x_1, x_2, \ldots x_k$, where $x_i \in (2i-2; 2i-1), i = 1, 2, \ldots, k$, and

$$P_{2k}(x_1) = P_{2k}(x_k) > P_{2k}(x_2) = P_{2k}(x_{k-1}) > \ldots > P_{2k}(x_{\frac{k}{2}}) = P_{2k}(x_{\frac{(k+2)}{2}}), \tag{27}$$

if $k$ is even, and

$$P_{2k}(x_1) = P_{2k}(x_k) > P_{2k}(x_2) = P_{2k}(x_{k-1}) > \ldots > P_{2k}(x_{\frac{(k+1)}{2}}), \tag{28}$$

if $k$ is odd.

Let us show now that the proposition 2 is true if $n = k+1$, i.e., for the points of maxima $x_1^{'}, x_2^{'}, \ldots, x_{k+1}^{'}$ of the polynomial

$$P_{2(k+1)}(x) = -x(x-1)(x-2)\ldots(x-(2k+1)), \tag{29}$$

where $x_i^{'} \in (2i-2; 2i-1), i = 1, 2, \ldots, k+1$, the following relation takes place:

$$P_{2(k+1)}(x_1^{'}) = P_{2(k+1)}(x_{k+1}^{'}) > P_{2(k+1)}(x_2^{'}) = P_{2(k+1)}(x_k^{'}) > \ldots > P_{2(k+1)}(x^{'}_{\frac{(k+2)}{2}}), \tag{30}$$

if $k$ is even, and

$$P_{2(k+1)}(x_1^{'}) = P_{2(k+1)}(x_{k+1}^{'}) > P_{2(k+1)}(x_2^{'}) = P_{2(k+1)}(x_k^{'}) > \ldots > P_{2(k+1)}(x^{'}_{\frac{(k+2)}{2}}) = P_{2(k+1)}(x^{'}_{\frac{(k+3)}{2}}), \tag{31}$$

if $k$ is odd.

Let us represent the polynomial $P_{2(k+1)}(x)$ as a complex polynomial:

$$P_{2(k+1)}(x) = P_{2k}(x) \cdot Q_2(x), \tag{32}$$

$$Q_2(x) = (x - 2k)(x - (2k + 1)) \tag{33}$$

where $Q_2(x)$ on the segment $(-\infty; 2k + \frac{1}{2})$ is a monotonically decreasing function. Now compare values of the polynomial $P_{2(k+1)}(x)$ in the neighbouring points of maxima $x_i^{'} \in (2i-2; 2i-1)$ and $x_{i+1}^{'} \in (2i; 2i+1)$ ($i = 1, 2, \ldots, \frac{k}{2,}$ if $k$ is even, and $i = 1, 2, \ldots, \frac{(k-1)}{2,}$ if $k$ is odd):

$$P_{2(k+1)}(x_i^{'}) = \max_{x\in(2i-2;2i-1)} P_{2(k+1)}(x) \geq P_{2(k+1)}(x_i) = P_{2k}(x_i) \cdot Q_2(x_i) > P_{2k}(x_i) \cdot Q_2(2i-1), \tag{34}$$

$$P_{2(k+1)}(x_{i+1}^{'}) = P_{2k}(x_{i+1}^{'}) \cdot Q_2(x_{i+1}^{'}) \leq \max_{x\in(2i;2i+1)} P_{2k}(x) \cdot Q_2(x_{i+1}^{'}) = P_{2k}(x_{i+1}) \cdot Q_2(x_{i+1}^{'}) < P_{2k}(x_{i+1}) \cdot Q_2(2i). \tag{35}$$

We obtain

$$P_{2(k+1)}(x_i^{'}) > P_{2k}(x_i) \cdot Q_2(2i-1) > P_{2k}(x_{i+1}) \cdot Q_2(2i) \geq P_{2(k+1)}(x_{i+1}^{'}), \tag{36}$$

i.e.,

$$P_{2(k+1)}(x_i^{'}) > P_{2(k+1)}(x_{i+1}^{'}), \tag{37}$$

where $i = 1, 2, \ldots, \frac{k}{2,}$ if $k$ is even, and $i = 1, 2, \ldots, \frac{(k-1)}{2,}$ if $k$ is odd.

By symmetry of the graph of the function with respect to the line $x = k + \frac{1}{2,}$ in other points of maxima of the polynomial $P_{2(k+1)}(x)$ we obtain necessary correlation.

We would like to get now a polynomial $Q_{2n}(x)$ with optimal distribution of maxima.

Let us change one of zeros of the polynomial $P_{2n}(x)$ to sufficiently small $\varepsilon$:

$$Q_{2n}(x) = -x(x-1)(x-2)\ldots(x-(n-1+(-1)^n\varepsilon))\ldots(x-(2n-1)). \tag{38}$$

Sign of the difference

$$Q_{2n}(x) - P_{2n}(x) = (-1)^n \varepsilon x(x-1)(x-2)\ldots(x-(n-2))(x-n)\ldots(x-(2n-1)) \tag{39}$$

on the segments between zeros $0, 1, 2, \ldots, n-2, n, \ldots, 2n-1$ of this difference will alternate increasing the value of the polynomial $Q_{2n}(x)$ comparing to the values of the polynomial $P_{2n}(x)$ in points of maxima on one side of the line $x = n - \frac{1}{2}$, and decreasing on the other.

*Proposition 3.*

The polynomial

$$Q_{2n}(x) = -x(x-1)(x-2)\ldots(x-(n-1+(-1)^n\varepsilon))\ldots(x-(2n-1)) \tag{40}$$

an arbitrary selected even degree $2n$ using condition $n \geq 3$, where $\varepsilon > 0$ is sufficiently small, has $n$ points of maxima $x_1^{'}, x_2^{'}, \ldots, x_n^{'}, x_i^{'} \in (2i-2; 2i-1), i = 1, 2, \ldots, n$, and

$$Q_{2n}(x_1^{'}) > Q_{2n}(x_n^{'}) > Q_{2n}(x_2^{'}) > Q_{2n}(x_{n-1}^{'}) > \ldots > Q_{2n}(x_{n/2}^{'}) > Q_{2n}(x_{(n+2)/2}^{'}), \tag{41}$$

if $n$ is even, and respectively

$$Q_{2n}(x_1^{'}) > Q_{2n}(x_n^{'}) > Q_{2n}(x_2^{'}) > Q_{2n}(x_{n-1}^{'}) > \ldots > Q_{2n}(x_{(n+1)/2}^{'}), \tag{42}$$

if $n$ is odd.

## Conclusion

We consider the acquired polynomial $Q_{2n}(x)$ as a function $G(x) = \int_0^x g(s)ds$, which is a primitive of the function $g(x)$. There exist exactly $(n-2)$ regular pairs of maxima points of the function $G(x)$, and, consequently, $(n-2)$ nontrivial period annuli of differential equation $x^{''} + g(x) = 0$.

## References

1. S. Atslega, F. Sadyrbaev. Period annuli in the Lienard type equation. – *Intern. J. of Pure and Applied Mathematics* 44(1) (2008) 117-123.
2. Y. Kozmina, F. Sadyrbaev. On a Maximal Number of Period Annuli. – *Abstract and Applied Analysis* Vol. 2011 (2011), Article ID 393875, 8 pages, doi:10.1155/2011/393875.
3. Y. Kozmina. Profiles of primitive functions and application to nonlinear differential equations. – *Information Technologies, Management and Society* 3(1) (2010) 64-67.